

SU(2) INVARIANTS OF SYMMETRIC QUBIT STATES

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Abstract

Density matrix for N-qubit symmetric state or spin-j state ($j = N/2$) is expressed in terms of the well known Fano statistical tensor parameters. Employing the multiaxial representation [1], wherein a spin-j density matrix is shown to be characterized by $j(2j+1)$ axes and $2j$ real scalars, we enumerate the number of invariants constructed out of these axes and scalars. These invariants are explicitly calculated in the particular case of pure as well as mixed spin-1 state.

Keywords: SU(2)Invariants, symmetric state, density matrix, quantum entanglement.

1 Introduction

The problem of enumeration of local invariants of quantum state described by a density matrix ρ is important in the context of quantum entanglement. Nonlocal correlations in quantum systems reflect entanglement between its parts. Genuine non- local properties should be described in a form invariant under local unitary operations. Two N-qubit states are said to be locally equivalent if one can be transformed into the other by local operations. i.e., $\rho' = U\rho U^\dagger$ where $U \in SU(2)^{\times N}$ and the two quantum states ρ and ρ' are said to be equally entangled.

A general prescription to identify the invariants associated with a multiparticle system have been outlined by Linden et al., [2]. Well known algebraic methods for generating invariants already exists in literature [3–6]. Williamson et al., [7] have presented a geometric approach for constructing SU(2) and SL(2,C) invariants. Makhlin [8] has presented a complete set of 18 local polynomial invariants of two qubit mixed states and demonstrated the usefulness of these invariants to study entanglement. As the number of subsystems increases, the problem of identifying and interpreting independent invariants rapidly becomes very complicated. Usha et al., [9] have shown that a set of 6 invariants which is a subset of a more general set of 18 invariants proposed by Makhlin [8] is sufficient to characterize the non local properties of a symmetric two qubit system. We also focus on symmetric two qubit states as the problem of identifying independent invariants become easier. Our approach makes use of the geometrical multiaxial representation of an arbitrary spin-j density matrix [1] which is completely characterized by a set of $j(2j+1)$ axes and $2j$ real positive scalars.

The paper is organized as follows: In Sec. 2, the decomposition of a density matrix in terms of well known Fano statistical tensor parameters is presented. Discussion of multi-axial description of the density matrix using Wigner-D matrices is outlined and the invariants associated with N-qubit symmetric state are enumerated in section 3. In Sec. 4, We have explicitly calculated the invariants of two qubit symmetric mixed as well as the most general pure state. To make our task easier, we have considered the Special Lakin Frame which is widely used in nuclear reactions.

2 Symmetric subspace

Here we are interested in the set of N-particle pure states that remain unchanged by permutations of individual particles. Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from 2^N to $(N + 1)$, when N qubits respect exchange symmetry. Such a Hilbert space is considered to be spanned by the eigen states $\{|j, m\rangle; -j \leq m \leq +j\}$ of angular momentum operators J^2 and J_z , where $j = \frac{N}{2}$. Analyzing a general state of N-particle spin-1/2 system represented by the density matrix of dimension $2^N \times 2^N$ is difficult because the system's Hilbert space increases exponentially with the number of qubits N. Fortunately, a large number of experimentally relevant states possesses symmetry under particle exchange and this property allows us to significantly reduce the computational complexity. Completely symmetric systems are experimentally interesting, largely because it is often easier to nonselectively address an entire ensemble of particles rather than individually address each member and it is possible to express the dynamics of these systems using only symmetry preserving operators. The symmetric subspace therefore provides a convenient, computationally accessible class of spin states. Specifically, if we have N two level atoms, each atom may be represented as a spin- 1/2 system and theoretical analysis can be carried out in terms of collective spin operator $\vec{J} = \frac{1}{2} \sum_{\alpha=1}^N \vec{\sigma}_\alpha$. Here $\vec{\sigma}_\alpha$ denote the Pauli spin operator of the α th qubit. The standard expression for the most general spin-j density matrix in terms of Fano statistical tensor parameters t_q^k 's is given by

$$\rho(\vec{J}) = \frac{Tr(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t_q^k \tau_q^{k\dagger}(\vec{J}), \quad (1)$$

where τ_q^k (with $\tau_0^0 = I$, the identity operator) are irreducible tensor operators of rank 'k' in the $2j+1$ dimensional spin space with projection 'q' along the axis of quantization in the real 3-dimensional space. The τ_q^k satisfy the orthogonality relations

$$Tr(\tau_q^{k\dagger} \tau_{q'}^k) = (2j+1) \delta_{kk'} \delta_{qq'}. \quad (2)$$

Here the normalization has been chosen so as to be in agreement with Madison convention [10]. The spherical tensor parameters t_q^k 's which characterize the given system are the average expectation values given by $t_q^k = \frac{Tr(\rho \tau_q^k)}{Tr \rho}$. Since ρ is Hermitian and $\tau_q^{k\dagger} = (-1)^q \tau_{-q}^k$, t_q^k satisfy the condition

$$t_q^{k*} = (-1)^q t_{-q}^k. \quad (3)$$

The spherical tensor parameters t_q^k 's have simple transformation properties under co-ordinate rotation in the 3-dimensional space. In the rotated frame t_q^k 's are given by

$$(t_q^k)^R = \sum_{q'=-k}^{+k} D_{q'q}^k(\phi, \theta, \psi) t_{q'}^k, \quad (4)$$

where $D_{q'q}^k(\phi, \theta, \psi)$ denote Wigner-D rotation matrix described by Euler angles (ϕ, θ, ψ) .

3 Multiaxial description of density matrix

It has already been shown [1] that a spin-j density matrix is characterized by $j(2j+1)$ axes and $2j$ real positive scalars. For the sake of completeness, we reproduce it here; In general $t_{\pm k}^k$ can be made zero for any k by suitable rotation. i.e.,

$$(t_{\pm k}^k)^R = 0 = \sum_{q'=-k}^{+k} D_{q',\pm k}^k(\phi, \theta, \psi) t_{q'}^k. \quad (5)$$

Using the well known Wigner expression for the rotation matrix D^k , the above equation can be written as

$$(t_{\pm k}^k)^R = 0 = [\pm_{\cos}^{sin}(\theta/2)]^{2k} \exp[i(\phi + \psi)] \sum_{r=0}^{2k} C_r Z^r, \quad (6)$$

where the complex variable $Z = \cot(\theta/2)e^{-i\phi}$ in the case of $(t_{+k}^k)^R = 0$ and $Z = \tan(\theta/2)e^{-i(\phi+\pi)}$ in the case of $(t_{-k}^k)^R = 0$. The expansion coefficients C_r in the polynomial are the same in both the cases and is

given by $C_r = \binom{2k}{k+q}^{\frac{1}{2}} t_q^k = \binom{2k}{r}^{\frac{1}{2}} t_{r-k}^k$. By solving the above polynomial equation, one can get

in general two sets of k-coordinate frames in which $(t_{\pm k}^k) = 0$. Explicitly, if $t_k^k = 0$ in a coordinate system where \hat{Z} axis is directed along (θ, ϕ) in the laboratory, $t_{-k}^k = 0$ in a coordinate system where \hat{Z} axis is directed along $(\pi - \theta, \phi + \pi)$. One set is obtained by the other by inverting the \hat{Z} -axis. Therefore it is sufficient to enumerate the k independent solutions $\hat{Q}_i(\theta_i, \phi_i)$, $i=1,2,\dots,k$ which constitute any arbitrary t_q^k as a spherical tensor product of the form

$$t_q^k = r_k(\dots((\hat{Q}_1 \otimes \hat{Q}_2)^2 \otimes \hat{Q}_3)^3 \otimes \dots)^{k-1} \otimes \hat{Q}_k)_q^k, \quad (7)$$

where

$$(\hat{Q}_1 \otimes \hat{Q}_2)_q^2 = \sum_{q_1} C(11k; q_1 q_2 q)(\hat{Q}_1)_{q_1}(\hat{Q}_2)_{q_2}; \quad (\hat{Q})_q = \sqrt{\frac{4\pi}{3}} Y_{1q}(\theta, \phi). \quad (8)$$

Here $C(11k; q_1 q_2 q)$ is the Clebsch Gordan Co-efficient and $Y_{1q}(\theta, \phi)$ are the well known spherical harmonics. If one of the \hat{Q}_i 's is inverted, sign of equation(7) is changed. Hence it is possible to choose k axes \hat{Q}_i 's, $i=1,2,\dots,k$ in such a way that r_k is always positive. Each axis requires two independent parameters (θ, ϕ) to characterize it, hence the k axes together with the overall multiplicative factor account for exactly $(2k+1)$ real parameters needed to specify a spherical tensor t_q^k satisfying equation (3). Thus any spherical tensor of rank k can be represented geometrically by a set of k vectors \hat{Q}_i on the surface of a sphere of radius r. Consequently, the state of a spin-j assembly can be represented geometrically by a set of $2j$ spheres, one corresponding to each value of k, $k=1,\dots,2j$, the kth sphere having k vectors specified on its surface.

Since $(\hat{Q}_i(\theta_i, \phi_i) \otimes \hat{Q}_j(\theta_j, \phi_j))_0^0$ is an invariant ($i \neq j$), one can construct in general $\binom{j(2j+1)}{2}$ invariants from $j(2j+1)$ axes. Together with $2j$ real positive scalars, there are $\binom{j(2j+1)}{2} + 2j$ invariants

characterizing spin-j density matrix. Thus using this multi-axial parametrization of density matrix, we enumerate the total number of SU(2) invariants characterizing a spin-j density matrix. Let us consider the example of two qubit symmetric state for a detailed discussion.

4 Invariants of two qubit symmetric state or spin-1 state

4.1 Pure spin-1 state

Consider the direct product $|\psi_1\rangle \otimes |\psi_2\rangle$ of two spinors in the qubit basis as

$$|\psi_{12}\rangle = \begin{pmatrix} \cos\frac{\theta_1}{2} \\ \sin\frac{\theta_1}{2}e^{i\phi_1} \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2}e^{i\phi_2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\phi_2} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\phi_1} \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i(\phi_1+\phi_2)} \end{pmatrix}, \quad (9)$$

$0 \leq \theta_{1,2} \leq \pi$, $0 \leq \phi_{1,2} \leq 2\pi$. In the symmetric angular momentum subspace $|11\rangle$, $|10\rangle$, $|1-1\rangle$, the combined state will have the form

$$|\psi_{12}\rangle_{sym} = \begin{pmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \frac{1}{\sqrt{2}}(\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\phi_2} + \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\phi_1}) \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i(\phi_1+\phi_2)} \end{pmatrix}. \quad (10)$$

Since the two directions (θ_1, ϕ_1) , (θ_2, ϕ_2) associated with the above two spinors define a plane, we choose this to be the xz-plane with respect to a frame $x_0y_0z_0$ with \hat{z}_0 being the bisector of the above two directions. Thus the azimuths of the above two directions (θ_1, ϕ_1) , (θ_2, ϕ_2) with respect to x_0 are respectively 0 and π . If the angular separation between the two directions is 2θ , then the state $|\psi\rangle$ has the explicit form

$$|\psi\rangle = \frac{\sqrt{2}}{\sqrt{1+\cos^2\theta}} [\cos^2\frac{\theta}{2}|11\rangle_{\hat{z}_0} - \sin^2\frac{\theta}{2}|1-1\rangle_{\hat{z}_0}]. \quad (11)$$

The density matrix corresponding to the above state is given by

$$\rho_s = \frac{2}{(1+\cos^2\theta)} \begin{pmatrix} \cos^4\frac{\theta}{2} & 0 & -\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \\ 0 & 0 & 0 \\ -\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} & 0 & \sin^4\frac{\theta}{2} \end{pmatrix}. \quad (12)$$

Comparing equation (12) with the standard representation of the density matrix

$$\rho_s = \frac{Tr(\rho)}{3} \begin{pmatrix} 1 + \sqrt{\frac{3}{2}}t_0^1 + \frac{t_0^2}{\sqrt{2}} & \sqrt{\frac{3}{2}}(t_{-1}^1 + t_{-1}^2) & \sqrt{3}t_{-2}^2 \\ -\sqrt{\frac{3}{2}}(t_1^1 + t_1^2) & 1 - \sqrt{2}t_0^2 & \sqrt{\frac{3}{2}}(t_{-1}^1 - t_{-1}^2) \\ \sqrt{3}t_2^2 & -\sqrt{\frac{3}{2}}(t_1^1 - t_1^2) & 1 - \sqrt{\frac{3}{2}}t_0^1 + \frac{t_0^2}{\sqrt{2}} \end{pmatrix}, \quad (13)$$

we get the non-zero t_q^k 's to be

$$t_0^1 = \frac{\sqrt{6}\cos\theta}{1+\cos^2\theta}, \quad t_0^2 = \frac{1}{\sqrt{2}}, \quad t_2^2 = t_{-2}^2 = \frac{\sqrt{3}\sin^2\theta}{2(1+\cos^2\theta)}.$$

Since $t_{\pm 1}^1 = 0$, \hat{z}_0 itself is the axis (\hat{Q}_1) associated with t^1 . As $t_0^1 = r_1(\hat{Q}_1)_0^1$,

$$r_1 = \frac{t_0^1}{(\hat{Q}_1)_0^1} . \quad (14)$$

Solving for the polynomial equation (6) for t^2 , we get $(\hat{Q}_2)_q^1 = \sqrt{\frac{4\pi}{3}}Y_q^1(\theta, 0)$ and $(\hat{Q}_3)_q^1 = \sqrt{\frac{4\pi}{3}}Y_q^1(\theta, \pi)$. Hence

$$r_2 = \frac{t_0^2}{(\hat{Q}_2 \otimes \hat{Q}_3)_0^2} = \frac{t_2^2}{(\hat{Q}_2 \otimes \hat{Q}_3)_2^2} . \quad (15)$$

The invariants associated with the most general pure spin-1 state are

$$I_1 = r_1 , I_2 = r_2 , I_3 = (\hat{Q}_1 \otimes \hat{Q}_2)_0^0 , I_4 = (\hat{Q}_1 \otimes \hat{Q}_3)_0^0 , I_5 = (\hat{Q}_2 \otimes \hat{Q}_3)_0^0 . \quad (16)$$

Explicitly,

$$I_1 = \frac{\sqrt{6}|\cos\theta|}{1 + \cos^2\theta} , I_2 = \frac{\sqrt{3}}{1 + \cos^2\theta} , I_3 = I_4 = -\frac{\cos\theta}{\sqrt{3}} , I_5 = -\frac{\cos 2\theta}{\sqrt{3}} . \quad (17)$$

It is clear from equation (11) that the state $|\psi\rangle$ is separable for $\theta = 0$ and π . Hence the invariants in the case of pure spin-1 separable states are

$$I_1 = \sqrt{\frac{3}{2}} , I_2 = \frac{\sqrt{3}}{2} , I_3 = I_4 = \mp \frac{1}{\sqrt{3}} , I_5 = -\frac{1}{\sqrt{3}} .$$

4.2 Mixed spin-1 state

Consider the example of channel spin-1 system which plays an important role in nuclear physics experiments like hadron scattering and reaction processes [11–15]. A beam of nucleons colliding with a proton target provides such an example. If both the beam and the target are prepared to be in mixed states, then the corresponding density matrices are given by

$$\rho(i) = \frac{1}{2} [I + \vec{\sigma}(i) \cdot \vec{p}(i)] = \frac{1}{2} \sum_{k,q} t_q^k(i) \tau_q^{k\dagger}(i); \quad i = 1, 2. \quad (18)$$

where $\vec{p}(i)$ are the polarization vectors and $\vec{\sigma}(i)$'s are the Pauli spin matrices.

The combined density matrix is the direct product of the individual density matrices

$$\rho_c = \rho(1) \otimes \rho(2) . \quad (19)$$

Eventhough the combined density matrix is a direct product of individual matrices, in this case, entanglement appears due to the projection of the combined density matrix onto the desired spin-1 space. While solving this problem the Special Lakin frame (SLF) which is widely used in studying nuclear reactions is considered : Choose \hat{z}_0 to be along $\vec{p}(1) + \vec{p}(2)$. Since $\vec{p}(1)$, $\vec{p}(2)$ together define a plane in any general situation, we choose \hat{x}_0 to be in this plane such that the azimuths of $\vec{p}(1)$, $\vec{p}(2)$ with respect to \hat{x}_0 are respectively 0 and π . The \hat{y}_0 axis is then chosen to be along $\hat{z}_0 \times \hat{x}_0$. The frame so chosen is indeed the special Lakin frame (SLF) as here $t_{\pm 1}^1 = 0$ and $t_2^2 = t_{-2}^2$. Choose a simple case of $|\vec{p}(1)| = |\vec{p}(2)| = p$, then we get $t_{\pm 1}^2 = 0$ in SLF. The density matrix so obtained for spin-1 mixed system

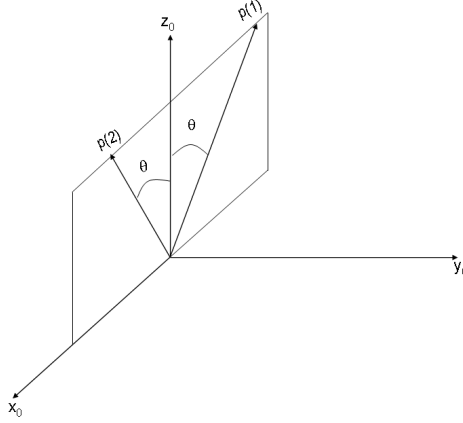


Figure 1: $x_0 y_0 z_0$ frame with mean spin direction \hat{z}_0 as the bisector of two directions $\vec{p}(1)$ and $\vec{p}(2)$

in the symmetric subspace $|11\rangle$, $|10\rangle$ and $|1-1\rangle$ is

$$\rho_s = \frac{1}{(3 + p^2 \cos 2\theta)} \begin{pmatrix} (1 + p \cos \theta)^2 & 0 & -p^2 \sin^2 \theta \\ 0 & 1 - p^2 & 0 \\ -p^2 \sin^2 \theta & 0 & (1 - p \cos \theta)^2 \end{pmatrix}. \quad (20)$$

Observe that when $p=1$, the mixed state density matrix is exactly the same as that of pure state density matrix as given by equation (12). Comparing the above density matrix with the standard form (equation (13)), we get the non-zero t_q^k 's as

$$t_0^1 = \frac{2\sqrt{6}p \cos \theta}{(3 + p^2 \cos 2\theta)}, t_0^2 = \frac{\sqrt{2}p^2(1 + \cos^2 \theta)}{(3 + p^2 \cos 2\theta)}, t_2^2 = \frac{\sqrt{3}p^2 \sin^2 \theta}{(3 + p^2 \cos 2\theta)}.$$

Solving for the polynomial equation (6) for t^1, t^2 , we obtain $\hat{Q}_1 = \hat{z}_0$, $\hat{Q}_2 = \vec{p}(1)$ and $\hat{Q}_3 = \vec{p}(2)$. Thus the invariants associated with the most general mixed spin-1 state are found to be

$$I_1 = \frac{2\sqrt{6}p|\cos \theta|}{(3 + p^2 \cos 2\theta)}, I_2 = \frac{2\sqrt{3}p^2}{(3 + p^2 \cos 2\theta)}, I_3 = I_4 = \cos \theta, I_5 = -\frac{\cos 2\theta}{\sqrt{3}}. \quad (21)$$

Note that in both pure as well as mixed state, $I_3 = I_4 = -\frac{\cos \theta}{\sqrt{3}}, I_5 = -\frac{\cos 2\theta}{\sqrt{3}}$. For $p = 1$ and $\theta = 0, \pi$, the state is separable as in the case of pure state. For $p < 1$, the state is separable for a range of values of θ . It is observed that as p decreases, the region of θ for which entanglement appears also decreases [16].

5 Conclusion

We have considered symmetric N-qubit density matrix expressed in terms of Fano statistical tensor parameters. Making use of the well known multi-axial decomposition of the density matrix we have enumerated SU(2) invariants of the most general symmetric state. Considering the special case of two qubit symmetric state we have explicitly computed 5 invariants which form a complete set. Our framework can be applied to enumerate a complete set of invariants of any qudit state. The study of the relationship between various measures of entanglement and our complete set of invariants is in progress.

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